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A Characterization of Tchebycheff Systems

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A characterization of Tchebycheff systems is given, in terms of Weak Tchebycheff systems.

Let M be a set of real numbers. A system $\{y_0, ..., y_n\}$ of real-valued functions defined on M is called a Tchebycheff system or T-system (Weak Tchebycheff system or WT-system), provided that M has at least $n + 1$ elements, and for every choice of points $t_0 < t_1 < \cdots < t_n$ of M, the determinant

$$D(y_0, ..., y_n/t_0, ..., t_n) = \det \| y_i(t_j); i, j = 0, ..., n \|$$

is strictly positive (nonnegative). If $\{y_0, ..., y_k\}$ is a T-system (WT-system) for $k = 0, ..., n$, then $\{y_0, ..., y_n\}$ is called a Complete Tchebycheff system or CT-system (Complete Weak Tchebycheff system or CWT-system). These definitions are consistent with the terminology employed in [1], but note that no assumptions of continuity have been made.

A system $\{y_0, ..., y_n\}$ of real-valued functions defined on M will be called "substantial," if for any interval (a, b), the functions $y_0, ..., y_n$ are linearly independent on $M \cap (a, b)$. In this paper we shall prove the following

THEOREM. *Let $\{y_0, ..., y_n\}$ be a system of real-valued functions defined on a dense subset M of an open interval. The following propositions are equivalent:*

(a) *The system $\{y_0, ..., y_n\}$ is a* T*-system on M.*

(b) *The system $\{y_0, ..., y_n\}$ is a substantial* WT*-system on M, and its linear span contains a function which does not vanish at any point of M.*

(c) *The system $\{y_0, ..., y_n\}$ is a substantial* WT*-system on M, and not all the functions y_i vanish simultaneously at any given point of M.*

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A particular case of this theorem was proved by Bartelt (cf. [2, Theorems 1 and 2]). Its proof will be carried out with the help of the following

LEMMA. *If* $\{y_0, ..., y_n\}$ *is a substantial* CWT-*system on a dense subset* M *of an open interval, and for some point* t_0 *of* M, $y_0(t_0) = 0$, *then* $y_r(t_0) = 0$ *for* $r = 0, 1, ..., n$.

Proof of Lemma. We proceed by induction on r. The assertion is true for $r = 0$ by hypothesis. Assume it to be true for $r \leqslant m$, and let $r = m + 1$. Since the system is substantial there exists a set $\{a_0, ..., a_m\}$, $a_0 < a_1 < \cdots < a_m < t_0$, of points of M, such that $D(y_0, ..., y_m/a_0, ..., a_m) = A > 0$.

Let $u_i(t) = D(y_0, ..., y_m/a_0, ..., a_{i-1}, t, a_{i+1}, ..., a_m)$. Thus $u_i(a_j) = 0$ if $i \neq j$, and $u_i(a_i) = A > 0$, whence

$$D(u_0, ..., u_m/a_0, ..., a_m) = \prod_{i=0}^{m} u_i(a_i) = A^{m+1} > 0. \tag{1}$$

It is now easy to see that $\{u_0, ..., u_m\}$ is a substantial WT-system on M. Indeed, since the column matrix $(u_j; j = 0, ..., m)$ admits of a representation of the form

$$(u_j; j = 0, ..., m) = Q \cdot (y_j, j = 0, ..., m),$$

where Q is the transition matrix, it is clear that for any choice $t_0 < \cdots < t_m$ of points of M,

$$D(u_0, ..., u_m/t_0, ..., t_m) = (\det Q) \cdot D(y_0, ..., y_m/t_0, ..., t_m), \tag{2}$$

In particular, setting $t_i = a_i$; $i = 0, ..., m$, we see from (1) and the definition of A, that $A^{m+1} = (\det Q) \cdot A$. Thus $\det Q = A^m > 0$, and the assertion readily follows from (2).

Since, as we have just seen, $\{u_0, ..., u_m\}$ is a substantial WT-system on M, and moreover $u_m \geqslant 0$ to the right of a_m, it is readily seen that there is a point t_1 of M, $t_0 < t_1$, such that $u_m(t_1) > 0$, i.e., $D(y_0, ..., y_m/a_0, ..., a_{m-1}, t_1) = B > 0$.

We are now ready to prove that $y_{m+1}(t_0) = 0$. In fact, since $y_i(t_0) = 0$; $i = 0, ..., m$, $0 \leqslant D(y_0, ..., y_{m+1}/a_0, ..., a_m, t_0) = A \cdot y_{m+1}(t_0)$. Since $A > 0$, $y_{m+1}(t_0) \geqslant 0$. On the other hand, $0 \leqslant D(y_0, ..., y_{m+1}/a_0, ..., a_{m-1}, t_0, t_1) = -B \cdot y_{m+1}(t_0)$. Since $B > 0$, $y_{m+1}(t_0) \leqslant 0$, and the conclusion follows. Q.E.D.

Proof of Theorem. The implication $a \Rightarrow b$ is a direct consequence of [4], Corollary 2. The implication $b \Rightarrow c$ being trivial, only $c \Rightarrow a$ remains to be proved.

We shall proceed by induction on n. The assertion is clearly true if $n = 0$. Assume it to be true for $n = m$, and let $n = m + 1$. Assume that there is a set $\{q_0, ..., q_{m+1}\}$ of points of M, $q_0 < \cdots < q_{m+1}$, such that $D(y_0, ..., y_{m+1}/q_0, ..., q_{m+1}) = 0$. Since the system is substantial, there exists a set $\{s_0, ..., s_{m+1}\}$ of points of M, with $q_{m+1} < s_0 < \cdots < s_{m+1}$, such that $A = D(y_0, ..., y_{m+1}/s_0, ..., s_{m+1}) > 0$. Defining $u_i(t) = D(y_0, ..., y_{m+1}/s_0, ..., s_{i-1}, t, s_{i+1}, ..., s_{m+1})$, we conclude, as in the proof of our Lemma, that $\{u_0, ..., u_{m+1}\}$ is a substantial WT-system on M, and a basis of the linear span of $\{y_0, ..., y_{m+1}\}$. Let $\{t_0, ..., t_i\}$ be a set of points of M such that $t_0 < \cdots < t_i < s_0$. Then

$$\begin{aligned} 0 &\leqslant D(u_0, ..., u_{m+1}/t_0, ..., t_i, s_{i+1}, ..., s_{m+1}) \\ &= \left[\prod_{j=i+1}^{m+1} u_j(s_j)\right] D(u_0, ..., u_i/t_0, ..., t_i) \\ &= A^{m+1-i} D(u_0, ..., u_i/t_0, ..., t_i). \end{aligned}$$

Since $A > 0$, we conclude that $\{u_0, ..., u_i\}$ is a substntial WT-system on the set of points of M to the left of s_0. Were u_0 to vanish at some point p_0 of this set, by the lemma, we would conclude that $u_i(p_0) = 0$; $i = 0, ..., m+1$. Thus all the functions y_i would vanish at p_0, in contradiction of (c). Hence, $u_0 > 0$ on the set of points of M to the left of s_0, and therefore the system $\{u_0, ..., u_m\}$ satisfies the conditions of (c) on $M \cap (-\infty, s_0)$. By inductive hypothesis, it is therefore a T-system thereon. It is also clear that $D(u_0, ..., u_{m+1}/q_0, ..., q_{m+1}) = 0$.

Consider now the function $y(t) = D(u_0, ..., u_{m+1}/t, q_1, ..., q_{m+1})$, which is clearly in the linear span of the system $\{u_0, ..., u_{m+1}\}$. The coefficient of u_{m+1} is $D(u_0, ..., u_m/q_1, ..., q_m) > 0$. Thus y is a nontrivial linear combination of the functions $u_0, ..., u_{m+1}$. Since these functions form a substantial system, it follows that there is a point t^* of M, $q_0 < t^* < q_1$, such that $y(t^*) > 0$, i.e., $D(u_0, ..., u_{m+1}/t^*, q_1, ..., q_m) > 0$. Let $q_0^* = t^*$, $q_i^* = q_i$, $i = 1, ..., m+1$, and define $v_i(t) = D(u_0, ..., u_{m+1}/q_0^*, ..., q_{i-1}^*, t, q_{i+1}^*, ..., q_{m+1}^*)$. Proceeding in the same way as for the functions u_i, it can be shown that $\{v_0, ..., v_{m+1}\}$ is a CWT-system on $M \cap (-\infty, t^*)$, and a basis of the linear span of $\{u_0, ..., u_{m+1}\}$. However, $v_0(q_0) = D(u_0, ..., u_{m+1}/q_0, ..., q_{m+1}) = 0$. Since $q_0 \in M \cap (-\infty, t^*)$, our lemma implies that all the functions v_i vanish at q_0. Since the functions v_i form a basis of the linear span of the functions y_i, we thus conclude that $y_i(q_0) = 0$, $i = 0, ..., m+1$, which contradicts the hypotheses of (c). Q.E.D.

Remark. Note that this theorem generalizes Theorem 3(b) of [5].

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